Quiz 6: 14.4-14.7i

Show all work clearly.

(1) Find all critical points of $f(x,y) = x^3 - y^2 - 4y + x^2 y$ and classify each as yielding local max., local min., or saddle points. Show how you arrive at your conclusions. You do not need to find the functional values at the critical points. Attach a computer graph that validates your answer.

(8 points)

Critical Points

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 + 2xy = 0 \\ -2y - 4 + x^2 = 0 \end{cases} \Rightarrow \begin{cases} x(3x+2y) = 0 \\ x^2 = 2y+4 \end{cases} \Rightarrow \begin{cases} \text{①}\ x = 0 \ \text{ or } \ \text{②}\ y = \frac{-3}{2}x \\ x^2 = 2y+4 \end{cases}$$

$$\text{①} \begin{cases} x = 0 \\ x^2 = 2y+4 \end{cases} \Rightarrow (0,-2)$$

$$\text{②} \begin{cases} y = \frac{-3}{2}x \\ x^2 = 2y+4 \end{cases} = x^2 = 2\left(-\frac{3}{2}x\right) + 4 \Rightarrow x^2 = -3x + 4$$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

$x = -4 \qquad x = 1$

$y = \frac{-3}{2}x \qquad y = 6 \qquad y = -3/2$

$(-4, 6) \qquad (1, -3/2)$

So critical points are $(0,-2)(-4,6)(1,-3/2)$

Second Derivative Test

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x+2y & 2x \\ 2x & -2 \end{vmatrix} = -2(6x+2y) - 4x^2$$

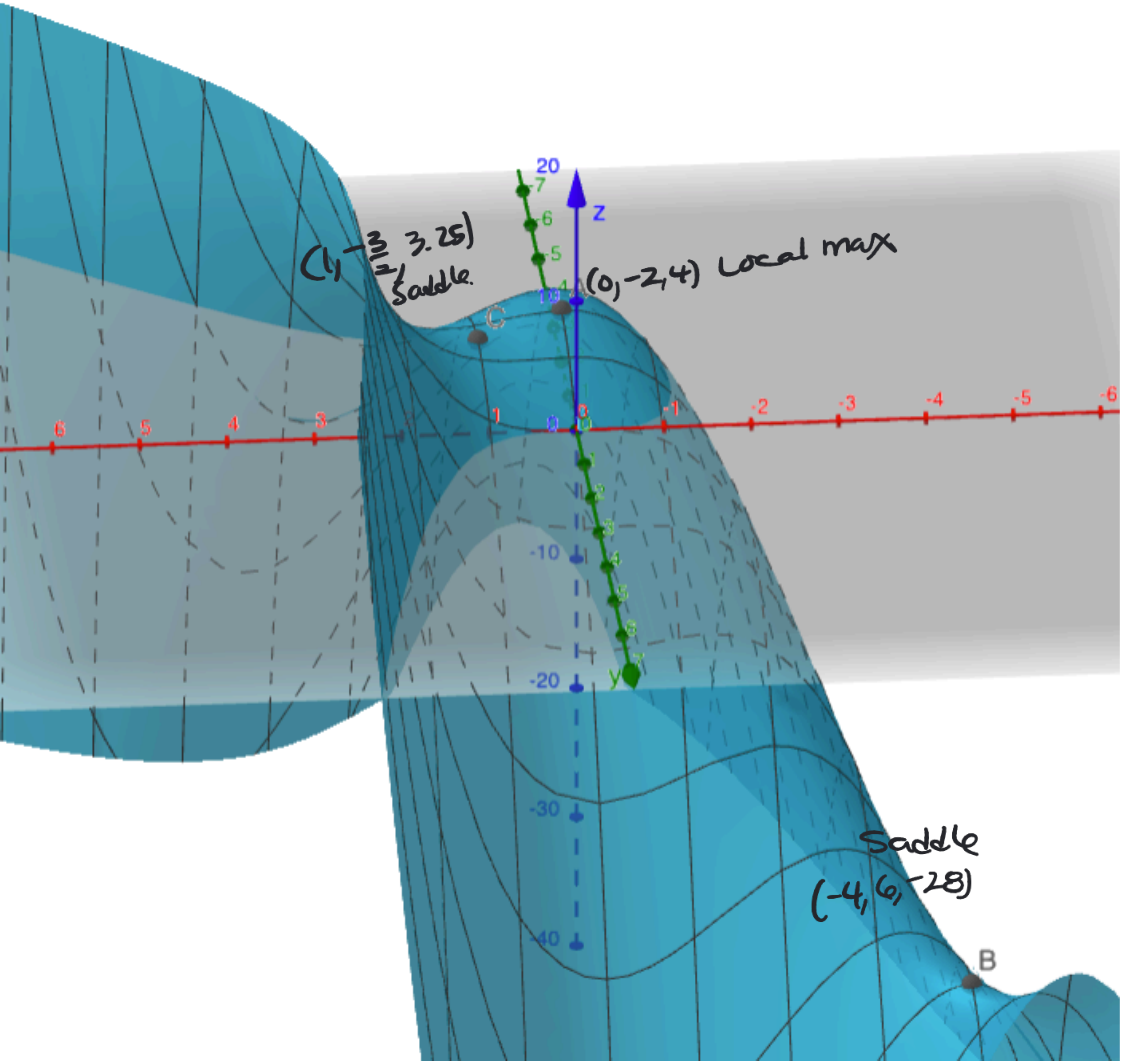
$$= -12x - 4y - 4x^2$$

$$= -4(3x + y + x^2)$$

$D(0,-2) = 8 > 0$, $f_{yy} < 0$: ∩ local max

$D(-4,6) = -40 < 0$ saddle point

$D(1,-3/2) = -10 < 0$ saddle



You should plot the points you get and set the scale so that all points are visible. The point is to see if your answer is reasonable

(2) Given $f(x,y)=\dfrac{x^3}{y^2+1}$, use differentials or a linear approximation to approximate the value of $f(1.01, 2.9)$ without using your calculator. (You can use your calculator to check your result).

(6 points)

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

where $f(a,b)$ is easily computed

$$f(1,3) = \frac{1}{10} \Rightarrow (a,b) = (1,3)$$

$$f_x = \frac{3x^2}{y^2+1} \qquad f_x(1,3) = \frac{3}{10}$$

$$f_y = \frac{-2x^3y}{(y^2+1)^2} \qquad f_y(1,3) = \frac{-6}{100}$$

$$\boxed{L(x,y) = \frac{1}{10} + \frac{3}{10}(x-1) - \frac{6}{10}(y-3)}$$

If you are using the tangent plane linear approx you should first find $L(x,y)$. Then use it to estimate $f(1.01, 2.9) \approx L(1.01, 2.9)$ by putting in 1.01 for x, 2.9 for y

$$f(1.01, 2.9) \approx L(1.01, 2.9)$$

$$= \frac{1}{10} + \frac{3}{10}(1.01-1) - \frac{6}{100}(2.9-3)$$

$$= .1 + .3(.01) - .06(-.1)$$

$$= .1 + .003 + .006 = \boxed{.109}$$

Can compare to calculator value $f(1.01, 2.9) = \dfrac{1.01^3}{2.9^2+1}$

$\approx .10949$

3) The temperature at a point (x, y, z) is $T(x,y,z) = 10z^3 e^{x^2 - \frac{y^2}{4}}$ degrees Celsius where x, y and z are measured in meters. *(6 points)*

Be sure to give appropriate units in answers.

(a) Find the rate of change of the temperature at (1,2,1) P in the direction toward (4,3,2) Q.

Direction $\vec{PQ} = \langle 3, 1, 1 \rangle$, need unit vector $u = \frac{1}{\sqrt{11}}\langle 3,1,1\rangle$

$$\vec{\nabla}T = \left\langle 10z^3 e^{x^2-\frac{y^2}{4}}(2x),\ 10z^3 e^{x^2-\frac{y^2}{4}} \cdot -\tfrac{1}{2}y,\ 30z^2 e^{x^2-\frac{y^2}{4}} \right\rangle$$

$$\vec{\nabla}T = 10e^{x^2-\frac{y^2}{4}} \left\langle 2xz^3, -\tfrac{1}{2}yz^3, 3z^2 \right\rangle$$

$$\vec{\nabla}T(1,2,1) = 10\langle 2, -1, 3\rangle$$

$$D_{\vec{u}}T(1,2,1) = 10\langle 2,-1,3\rangle \cdot \frac{1}{\sqrt{11}}\langle 3,1,1\rangle$$

$$= \frac{10}{\sqrt{11}}(6-1+3) = \frac{80}{\sqrt{11}} \ {}^\circ C/m$$

(b) A bug at (1,2,1) wants to fly in the direction in which the temperature increases most rapidly. In what direction should the bug travel? What is the rate of increase in that direction? (units)

In direction of $\vec{\nabla}T(1,2,1) = \langle 20, -10, 30\rangle$

Rate of change is $\|\vec{\nabla}T(1,2,1)\| = 10\sqrt{14}\ {}^\circ C/m$

Note: If you are just specifying a direction, it does not have to be a unit vector.

It is only if you are using it to find the derivative $D_{\vec{u}}f$ that you need a unit vector.